CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
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CIP seminar, Friday conversations:,
For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks.

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
   1. w.r.t. a functor with - at least - two combinatorial applications:
      1. the two routes to reach the free algebra
      2. alphabets interpolating between commutative and non commutative worlds
   2. without functor: sums, tensor and free products
   3. w.r.t. a diagram: colimits
3. Representation theory.
4. MRS factorisation: A local system of coordinates for Hausdorff groups and fine tuning between analysis and algebra.
5. This scope is a continent and a long route, let us, today, walk part of the way together.

Disclaimers.

Disclaimer I.– The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

Disclaimer II.– Sometimes, absolute rigour is not followed[a]. In its place, from time to time, we will seek to give the reader an intuitive feel for what the concepts of category theory are and how they relate to our ongoing research within CIP, CAP and CCRT.

[a] All is assumed to be subsequently clarified on request though.

Disclaimer III.– The reader will find repetitions and reprises from the preceding CCRT[n], they correspond to some points which were skipped or uncompletely treated during preceding seminars.

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Matters

1. Representation theory (or theories)
    1. Geometric point of view
    2. Combinatorial point of view (Ram and Barcelo manifesto)
    3. Categorical point of view
2. From groups to algebras
   Here is a bit of rep. theory of the symmetric group, deformations, idempotents
3. Irreducible and indecomposable modules
4. Characters, central functions and shifts.
   Here are (some of) Lascoux and Schützenberger's results
5. Reductibility and invariant inner products
   Here stands Joseph's result
6. Commutative characters
   Here are time-ordered exponentials, iterated integrals, evolution equations and Minh's results
7. Lie groups Cartan theorem
   Here is BTT

CCRT[28] Presentations and adjunctions.

Plan of this talk.

1. The categories of this talk.
2. Universal problems
   1. wrt a functor
   2. wrt a very special diagram, leading to coequalization of presentations
3. Mixing both
4. (D) and (LF) monoids (opt.)
5. Application to $S' = MS$ and Picard (opt.)
6. Concluding remarks

Outline

Categories of this talk.

1. These categories are as follows
    1. **Set** the category of sets.
    2. **Mon**, the category of monoids.
    3. **k-Lie**, the category of **k**-Lie algebras.
    4. **Grp**, the category of groups.
    5. **k-AAU**, the category of **k** associative algebras with unit.
2. Functors are as follows

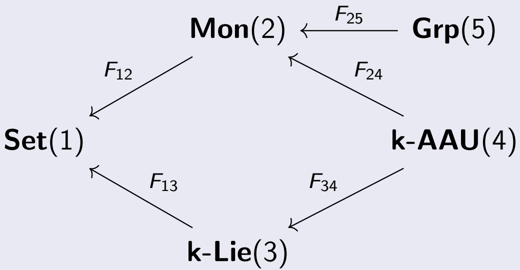


Figure: Rq: Similar lower diagram with **k-Mod** replacing **Set**.

Adjunction as free generation wrt a functor: general principle.

1. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories and $F: \mathcal{C}_2 \to \mathcal{C}_1$ a (covariant) functor between them

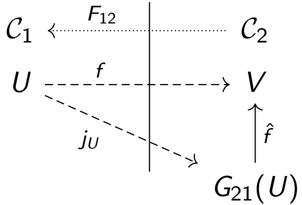


Figure: A solution of the universal problem (of free generation) w.r.t. the functor F. Note that this problem may have no solution (ex. $\mathcal{C}_1 =$ **FinSet** and $\mathcal{C}_2 =$ **FinMon**.)

$$\big(\forall f \in Hom(U, F[V])\big)\big(\exists! \, \hat{f} \in Hom(G_{21}(U), V)\big)\big(F(\hat{f}) \circ j_U = f\big)$$

Adjuncts for the F_{ij}.

1. $G_{21}(X) = X^*$, the free monoid
2. $G_{31}(X) = \mathcal{L}ie_{\mathbf{k}}(X)$, the free Lie algebra $(\subset \mathbf{k}\langle X \rangle)$
3. $G_{43}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra
4. $G_{42}(M) = \mathbf{k}[M]$, the $\mathbf{k}$-algebra of the monoid M
5. $G_{52}(M) = G(M)$, the universal group of the monoid M (see [16] VII.3). In the commutative case, it is called the Grothendieck group of M.

1. **Remark.–** There is a transfer of freeness (as a module) for G_{43} (Universal enveloping algebra) but nothing similar for G_{52} (Universal group).

What is a presentation ? Examples.

6. Let us first begin with **Mon** and **Grp**.
7. Our first example is D_n the dihedral group D_n

$$\langle\ r, s\ ;\ r^n = s^2 = (sr)^2 = 1\rangle_{\mathbf{Grp}} \tag{1}$$

(see next slide for $n = 5$)

8. Indeed a presention is always: (1) an alphabet (of generators), (2) a list (of relations, or relators), (3) a category (as index).
9. The Moore-Coxeter presentation of the symmetric group $\mathfrak{S}_n$. It reads

$$\begin{aligned}\langle\ (t_i)_{1\leq i\leq n-1}\ ; \qquad & t_it_{i+1}t_i = t_{i+1}t_it_{i+1}\ (i \leq n-2), \\ & t_it_j = t_jt_i\ (|i-j| \geq 2) \\ & t_i^2 = 1\ (i \leq n-1)\ \rangle_{\mathbf{Grp}}\end{aligned}$$

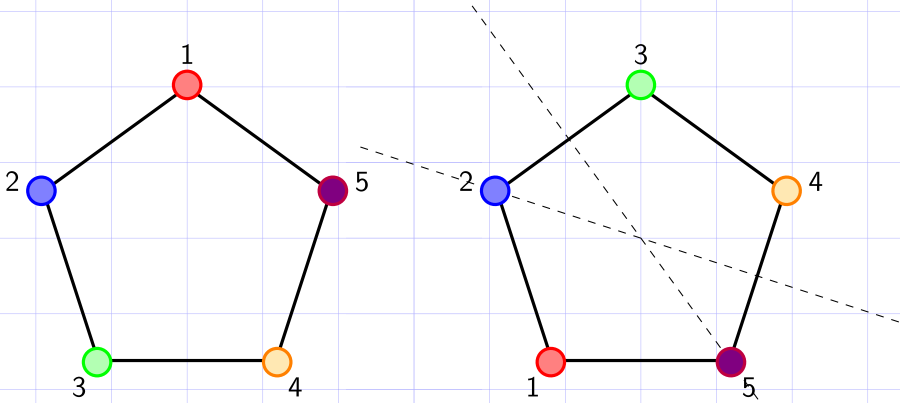


Figure: For D_5 (group of order 10). Coxeter presentation is with s_1 (symmetry wrt the line passing through node 5) and s_2 (symmetry wrt the line passing through node 2) and relator $[s_i^2 = 1 \; ; \; (s_1 s_2)^5 = 1]$.

Categorical setting for a presentation

10. The presented structure is a quotient of the free one $Free(X)$ (with the same set of relators): X^* for monoids (**Mon**), $F(X)$ for groups (**Grp**), $\mathcal{L}ie_{\mathbf{k}}(X)$ for **k**-Lie algebras (**k-Lie**), $\mathbf{k}\langle X\rangle$ for **k-AAU**.
11. The list of relators can be put in the form $(u_s = v_s)_{s\in T}$ where $u_s, v_s \in Free(X)$.
12. For example, the **k**-Drinfeld-Kohno Lie algebra of order n, DK_n is presented by a set of generators with symmetry $t_{ij} = t_{ji},\ 1 \leq i \neq j \leq n-1$ and

$$\langle\ (t_{ij})_{1\leq i\neq j\leq n-1}\ ;\ [t_{ij}, t_{kl}] = [t_{ij}, t_{ik} + t_{jk}] = 0\ |\{i,j,k,l\}| = 4\ \rangle_{\mathbf{k}\text{-}\mathbf{Lie}}$$

(see slide 15 below for precise conditions).

What does it mean to say that *the relators are in another category* ? (which is always a value)

13 In the literature and the discussions, one oftentimes finds "the relators are monoidal" or "are of Lie type". Let me give you three examples

14 First one, **the Braid group** B_n

$$\langle\, (t_i)_{1\leq i\leq n-1}\ ; \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}\ (i \leq n-2),$$
$$t_i t_j = t_j t_i\ (|i-j| \geq 2)\,\rangle_{\mathbf{Grp}}$$

15 In this case, the presentation is **monoidal**, one can define the monoid B_n^+ of "positive" (or simple) braids and we have $B_n = G(B_n^+)$, i.e. B_n is (isomorphic to) the universal group of the monoid B_n^+.

What does it mean to say that *the relators are in another category* ? (which is always a value)/2

16 The second example is given by the Drinfeld-Kohno algebra presented by a set of $(n-1)(n-2)$ generators and the relator

$$\mathbf{R} = \left\{ \begin{array}{rcll} t_{i,j} & = & t_{j,i} & \text{for distinct } i,j, \quad (i \neq j), \\ [t_{i,j}, t_{i,k} + t_{j,k}] & = & 0 & \text{for distinct } i,j,k, \quad (i \neq j \neq k), \\ [t_{i,j}, t_{k,l}] & = & 0 & \text{for distinct } i,j,k,l, \quad (i \neq j \neq k \neq l). \end{array} \right.$$

$$DK_{\mathbf{k}\text{-}\mathbf{AAU}}[n] = \langle\ (t_{ij})_{1 \leq i \neq j \leq n-1} \quad ; \quad \mathbf{R}\ \rangle_{\mathbf{k}\text{-}\mathbf{AAU}}$$

17 In this case, the presentation is **of Lie type**, one can define the Drinfeld-Kohno **k**-Lie algebra $DK_{\mathbf{k}\text{-}\mathbf{Lie}}[n]$ and we have automatically

$$DK_{\mathbf{k}\text{-}\mathbf{AAU}}[n] \simeq \mathcal{U}(DK_{\mathbf{k}\text{-}\mathbf{Lie}}[n])$$

What does it mean to say that *the relators are in another category* ? (which is always a value)/3

18 The third example is that of partially commutative structures [13]

19 They are given in the categories

Mon ; **Grp** ; **k-Lie** and **k-AAU**

by a set of commutations $xy = yx$ (or $[x, y] = 0$).

20 These presentations are **monoidal** or of **Lie type**.

21 This double-face is the core of the closure theorem.

Categorical setting for a presentation/2

22. For the considered categories, we have a forgetful functor $F : \mathcal{C} \to \mathbf{Set}$, and the following diagram

$$T \overset{u_\bullet}{\underset{v_\bullet}{\rightrightarrows}} Free(X) \tag{2}$$

23. The presented algebra and its arrow $Free(X) \xrightarrow{j} P$ is then a solution of the following universal problem

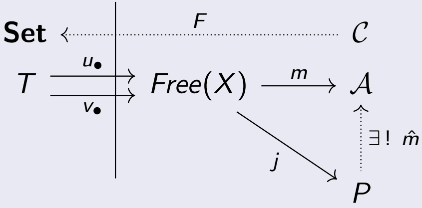


Figure: The arrow m is a morphism within the category $\mathcal{C}$ which equalizes the relators i.e. $F(m \circ u_\bullet) = F(m \circ v_\bullet)$. The arrow m is a coequalizer.

Categorical setting for a presentation: transitivity.

24. If the relator presenting P_2 is a set of a "lower category" The presented algebra and its arrow $Free(X) \xrightarrow{j} \mathcal{A}$ is then a solution of the following universal problem

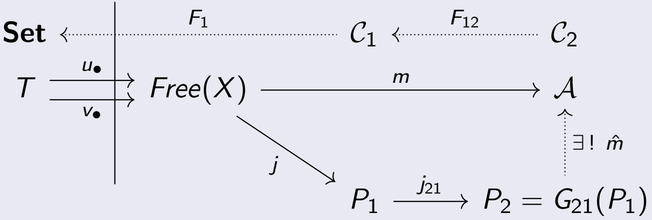


Figure: The arrow m is a morphism within the category $\mathcal{C}$ which equalizes the relators i.e. $F(m \circ u_\bullet) = F(m \circ v_\bullet)$. The arrow m is a coequalizer.

An example: structure of the DK algebra.

25 The noncommutative indeterminates $\{t_{i,j}\}_{1\leq i\neq j\leq n-1}$ are now bound by the following infinitesimal braid relations

$$\mathbf{R} = \left\{ \begin{array}{rcll} t_{i,j} & = & t_{j,i} & \text{for distinct } i,j, \quad (i\neq j), \\ [t_{i,j}, t_{i,k}+t_{j,k}] & = & 0 & \text{for distinct } i,j,k, \quad (i\neq j\neq k), \\ [t_{i,j}, t_{k,l}] & = & 0 & \text{for distinct } i,j,k,l, \quad (i\neq j\neq k\neq l). \end{array} \right. \tag{3}$$

26 Let $\mathcal{J}_n$ be the two-sided ideal of $\mathbf{k}\langle\mathcal{T}_n\rangle$ generated by these relations.

27 The $\mathbf{k}$-AAU presented as

$$\langle\ \mathcal{T}_n\ ;\ \mathbf{R}\rangle_{\mathbf{k}\text{-}\mathbf{AAU}} \tag{4}$$

is then $\mathcal{A}\langle\mathcal{T}_n\rangle\big/\mathcal{J}_n$

28 By transitivity

$$\mathcal{A}\langle\mathcal{T}_n\rangle\big/\mathcal{J}_n \simeq \mathcal{U}(DK_n)$$

Towards Gröbner bases: Splitting the DK Lie algebra (as a module).

29. Let A_i be the free Lie algebra generated by $\{t_{ij} \mid i < j \leq n-1\}$ i.e.

$$\begin{aligned} A_{n-2} &\text{ is generated by } \{t_{(n-2),(n-1)}\} \\ A_{n-3} &\text{ is generated by } \{t_{(n-3),(n-2)}, t_{(n-3),(n-1)}\} \end{aligned} \tag{5}$$

It can be shown that

$$L_n \simeq_{\text{ mod } \mathbf{k}} A_1 \oplus A_2 \oplus \cdots \oplus A_{n-1} \tag{6}$$

See [15] (A_i is an ideal of the sum $A_i \oplus \cdots \oplus A_{n-1}$, to check).

30. Algorithmically, this gives a pathway to concrete computation of bases (Lyndon, Hall, finely hompogeneous and then MRS).

Free partially commutative structures.

31. There is a set of structures (free partially commutative, see [13]).
32. Given a reflexive graph $\Delta_X \subset \theta \subset X$ (X is the alphabet)

$$M(X,\theta) = \langle X; (xy = yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}} \tag{7}$$

    where $\theta \subset X \times X$ is a reflexive undirected graph.
33. These structures are compatible with Lazard's elimmination and MRS factorization. This can proved using $\mathbf{k}[M(X,\theta)] = \mathcal{U}(Lie_{\mathbf{k}}(X,\theta))$.
34. A unipotent Magnus group with a nice Log-Exp correspondence can be defined more generally for every locally finite monoid. Is there a general MRS factorization ?
35. In the sound cases, what is the combinatorics of different orders ? (Not increasing or decreasing Lyndon words.) Are they useful ?

Ex1

36. The bicyclic monoid

$$By = \langle\ a, b\ ;\ ba = 1\ \rangle_{\mathbf{Mon}} \tag{8}$$

has a normal form $(a^p b^q)_{p,q\in\mathbb{N}}$ and then is not a group (otherwise, we would have $ab = 1$ which is not the case).

37. How to prove this ?
    1. One surmises the normal forms (here $\{a^p b^q\}_{p,q\in\mathbb{N}} = a^* b^*$ and one "abstracts" $L = a^p b^q$ in some way (here it will be $[p, q]$
    2. one defines the transitions i.e.

       $$a.[p, q] := [p+1, q]\ b.[p, q] := [p-1, q](p > 0) \text{ and } b.[0, q] := [0, q+1]$$

    3. One proves that $x \mapsto t_x$ is a representation of By in L^L (i.e. respects the defining relations.

Ex2

38. Monoids
$$\langle\ X\ ;\ (u_i = v_i)_{i\in I}\ \rangle_{\mathbf{Mon}}$$
with $|u_i| = |v_i|$ $(i \in I)$ are $\mathbb{N}$-graded
39. If, moreover, for all $x \in X$, we have $|u_i|_x = |v_i|_x$ (Schützenberger called them "commutation monoids"), then they are (LF). See below.
40. The Braid monoid (same presentation than the Braid group, but within the category **Mon**) is graded but NOT finely homogeneous.

Monoids and series (D) and (LF) monoids of monomials.

41 We recall here the discussion of [10] about monoids and series.

42 A set E being given, $\mathbf{k}^E$ is the set of all functions $f \in E \to \mathbf{k}$,

1. $\mathrm{supp}(f) = \{x \in E \mid f(x) \neq 0\}$
2. $\mathbf{k}^{(E)} = \{f \in \mathbf{k}^E \mid \#(\mathrm{supp}(f)) < \infty\}$
3. $\langle S|P\rangle = \sum_{x\in E} S(x)P(x),\ S \in \mathbf{k}^E,\ P \in \mathbf{k}^{(E)}$

43 Starting with a monoid $(M, ., 1_M)$ and considering $\mathbf{k}^{(M)} = \mathbf{k}[M] \subset \mathbf{k}[[M]] = \mathbf{k}^M$, we see that in order to extend the product formula

$$P.Q := \sum_{w\in M} \sum_{u.v=w} \langle P|u\rangle\langle Q|v\rangle\, w \qquad (9)$$

it is sufficient (and necessary in general position) that the map $\star : M \times M \to M$ has finite fibers[a] (condition [D], see [2] III §2.10).

[a] Recall that a map $f:\ X \to Y$ between two sets X and Y has finite fibers if and only if for each $y \in Y$, the preimage $f^{-1}(y)$ is finite.

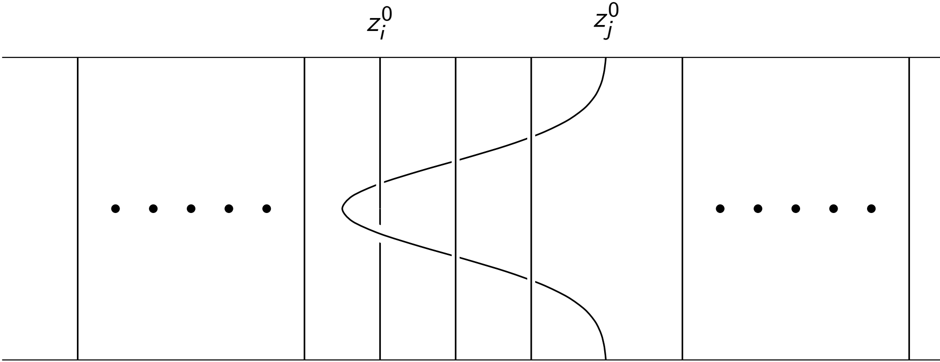
(D) and (LF) monoids/2

44. If M satisfies condition [D], we can extend the formula (9) to arbitrary $P, Q \in \mathbf{k}^M$ (as opposed to merely $P, Q \in \mathbf{k}[M]$). In this case, the $\mathbf{k}$-algebra $\left(\mathbf{k}^M, ., 1_M\right)$ is called the total algebra of M, [a] and its product is the Cauchy product between series.
45. For every $S \in \mathbf{k}^M$, the family $(\langle S|m\rangle\, m)_{m\in M}$ is summable[b]. and its sum is precisely $S = \sum_{m\in M}\langle S|m\rangle\, m$.

[a] See also https://en.wikipedia.org/wiki/Total_algebra.

[b] We say that a family $(f_i)_{i\in I}$ of elements of $\mathbf{k}^M$ is summable if for any given $m \in M$, all but finitely many $i \in I$ satisfy $\langle f_i|m\rangle = 0$. Such a summable family will always have a well-defined infinite sum $f = \sum_{m\in M}\sum_{i\in I}\langle f_i|m\rangle\, m \in \mathbf{k}^M$, whence the name "summable".

Braids



(D) and (LF) monoids/3

46 For example, the monoid $M = \{x^k\}_{k\in\mathbb{Z}}$, a multiplicative copy of $\mathbb{Z}$ does not satisfy condition [D].

47 Then, $\mathbf{k}[M] = \mathbf{k}[x, x^{-1}]$ is the algebra of Laurent polynomials. It admits no total algebra.

48 For this monoid, we have to impose a constraint of the support (i.e. admit only supports like $[a, +\infty[_{\mathbb{Z}}$. The resulting algebra, $\mathbf{k}[x, x^{-1}]]$ is that of Laurent series.

(D) and (LF) monoids/4

49 For every series $S \in \mathbf{k}[[M]]$, we set $S_+ := \sum_{m\neq 1} \langle S|m\rangle\, m$.
In order for the family $((S_+)^n)_{n\geq 0}$ to be summable, it is sufficient that the iterated multiplication map $\mu^* :\ (M_+)^* \to M$ defined by

$$\mu^*[m_1, \ldots, m_n] = m_1 \cdots m_n \text{ (product within } M) \tag{10}$$

have finite fibers (where we have written the word $[m_1, \ldots, m_n] \in (M_+)^*$ as a list to avoid confusion).[a]

50 In this case the characteristic series of M (i.e. $\underline{M} = \sum_{m\in M} m = 1 + \underline{M_+}$) is invertible and its inverse is called the Möbius function $\mu :\ M \to \mathbb{Z}$. It is such that

$$\underline{M}^{-1} = 1 - \underline{M_+} + \underline{M_+}^2 - \underline{M_+}^3 - \cdots = \sum_{m\in M} \mu(m).m \tag{11}$$

[a] Furthermore, this condition is also necessary (if S_+ is generic) if $\mathbf{k} = \mathbb{Z}$. These monoids are called "locally finite" in [18].

Examples and remarks

51 Every finite monoid (and in particular finite groups) satisfies condition (D).

52 Among finite groups, only the trivial group is locally finite.

53 Many combinatorial monoids are such that $M_+ = M \setminus \{1_M\}$ is stable by products.

54 For example X^*, $X^* \otimes X^*$ and $\mathbb{N}^{(X)}$ (the free abelian monoid)

55 In the case of point 53, $S \mapsto \langle S|1_M\rangle$ is a character of $\mathbf{k}[[M]]$ (with values in $\mathbf{k}$).

56 In the case of point 54, these monoids are locally finite, each M^{-1} is polynomial and given by, respectively

$$1 - X \;;\; 1 - \sum_{x\in X}(x \otimes 1 + 1 \otimes x) + \sum_{x,y\in X} x \otimes y \;;\; \prod_{x\in X}(1 - x) \qquad (12)$$

where $\mathbb{N}^{(X)}$ is written multiplicatively $\{X^\alpha\}_{\alpha\in\mathbb{N}^{(X)}}$.

Some concluding sentences and remarks

1. We have seen the notion of a left adjoint from the point of view of the universal problem consisting in the free generation wrt to a functor.
2. We have seen the coequalization problem as a particular case of a colimit universal problem, a diagram and a functor being given.
3. From this last point of view we get, for free, a transitivity criterium for presented structures with relators pertaining to another "forgetted" category.
4. In the forthcoming CCRT issues, we will see more examples of presented monoids and Lie algebras.

Thank you for your attention.

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